

# FUSION FRAMES AND RANDOMIZED SUBSPACE ACTIONS

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**ABSTRACT.** A randomized subspace action algorithm is investigated for fusion frame signal recovery problems. It is noted that Kaczmarz bounds provide upper bounds on the algorithm's error moments. The main question of which probability distributions on a random fusion frame lead to provably fast convergence is addressed. In particular, it is proven which distributions give minimal Kaczmarz bounds, and hence give best control on error moment upper bounds arising from Kaczmarz bounds. Uniqueness of the optimal distributions is also addressed.

## 1. INTRODUCTION

Fusion frames are a mathematical tool for distributed signal processing and data fusion. Complexity and computational constraints in high dimensional problems can limit the amount of global processing that is possible and often require approaches that are built up from local processing. For example, in wireless sensor networks, physical constraints on sensors mean that global processing is typically organized through processors on local subnetworks. Mathematically, a fusion frame provides global signal representations by fusing together projections on local subspaces.

Fusion frames were introduced in [8] as a generalization of the classical notion of frames. Let  $H$  be a Hilbert space and let  $I$  be an at most countable index set. A collection  $\{\varphi_n\}_{n \in I} \subset H$  is a *frame* for  $H$  with *frame bounds*  $0 < A \leq B < \infty$  if

$$\forall x \in H, \quad A\|x\|^2 \leq \sum_{n \in I} |\langle x, \varphi_n \rangle|^2 \leq B\|x\|^2. \quad (1.1)$$

The frame inequality (1.1) ensures that the frame coefficients  $\langle x, \varphi_n \rangle$  stably encode  $x \in H$ , and that there exists a (possibly nonunique) dual frame  $\{\psi_n\}_{n \in I} \subset H$  such that the following unconditionally convergent frame expansions hold

$$\forall x \in H, \quad x = \sum_{n \in I} \langle x, \varphi_n \rangle \psi_n = \sum_{n \in I} \langle x, \psi_n \rangle \varphi_n. \quad (1.2)$$

An important aspect of frame theory is that frames can be redundant or overcomplete. Redundancy endows the frame expansions (1.2) with robustness properties that are useful in applications such as multiple description coding [19], transmission of data over erasure

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channels [18, 20], and quantization [1]. See [4, 9] for an introduction to frame theory and its applications.

Fusion frames take the idea (1.1) one step further and replace the scalar frame coefficients  $\langle x, \varphi_n \rangle$  by projections onto a redundant collection of subspaces. Let  $\{W_n\}_{n \in I}$  be a collection of closed subspaces of  $H$  and let  $\{v_n\}_{n \in I} \subset (0, \infty)$  be a collection of positive weights. The collection  $\{(W_n, v_n)\}_{n \in I}$  is said to be a *fusion frame* for  $H$  with *fusion frame bounds*  $0 < A \leq B < \infty$  if

$$\forall x \in H, \quad A\|x\|^2 \leq \sum_{n \in I} v_n^2 \|P_{W_n}(x)\|^2 \leq B\|x\|^2, \quad (1.3)$$

where  $P_W$  denotes the orthogonal projection onto a subspace  $W \subset H$ . If  $A = B$  then the fusion frame is said to be *tight*. Note that in the special case when each  $W_n$  is a one-dimensional subspace with  $W_n = \text{span}(\varphi_n)$  and  $v_n = \|\varphi_n\|^2$ , the fusion frame inequality (1.3) reduces to the statement (1.1) that  $\{\varphi_n\}_{n \in I}$  is a frame for  $H$ . For further background on fusion frames see [6, 7, 3, 5].

If  $\{(W_n, v_n)\}_{n \in I}$  is a fusion frame for  $H$  then the associated *fusion frame operator*  $S : H \rightarrow H$  is defined by

$$S(x) = \sum_{n \in I} v_n^2 P_{W_n}(x).$$

It is known, e.g., [8, 10], that  $S$  is a positive invertible operator and that each  $x \in H$  can be recovered from its fusion frame projections  $y_n = P_{W_n}(x) \in H$ ,  $n \in I$ , by

$$x = (S^{-1} \circ S)(x) = \sum_{n \in I} v_n^2 (S^{-1} \circ P_{W_n})(x) = \sum_{n \in I} v_n^2 S^{-1}(y_n). \quad (1.4)$$

Practical inversion of the fusion frame operator  $S$  and, more generally, reconstructing  $x \in \mathbb{R}^d$  from the projections  $y_n = P_{W_n}(x)$ , can be computationally intensive.

The following extension of the classical frame algorithm gives an iterative way to recover  $x \in H$  from its fusion frame projections  $y_n = P_{W_n}(x)$ ,  $n \in I$ . Given an arbitrary initial estimate  $x_0 \in H$ , the algorithm produces estimates  $x_n \in H$  of  $x$  from the projections  $\{y_n\}_{n \in I}$  by iterating for  $n \geq 1$

$$x_n = x_{n-1} + \frac{2}{A+B} \left[ \left( \sum_{j \in I} v_j^2 y_j \right) - S(x_{n-1}) \right] = x_{n-1} + \frac{2}{A+B} S(x - x_{n-1}). \quad (1.5)$$

Similar to the situation for frames, it was shown in [10] that the fusion frame algorithm (1.5) satisfies  $\lim_{n \rightarrow \infty} x_n = x$  with

$$\|x - x_n\| \leq \left( \frac{B-A}{B+A} \right)^n \|x\|. \quad (1.6)$$

For other iterative approaches to fusion frame reconstruction and a discussion of local versus global aspects of reconstruction see [10, 22, 27].

**1.1. Recovery by iterative subspace actions.** This article will analyze a particular fusion frame recovery algorithm that is motivated by row-action methods. We henceforth restrict our attention to the finite dimensional setting and let  $\mathbb{H}^d$  be the  $d$ -dimensional Hilbert

space with  $\mathbb{H} = \mathbb{R}$  or  $\mathbb{C}$ . Suppose that  $\{W_n\}_{n=1}^\infty$  is a given collection of subspaces of  $\mathbb{H}^d$ . The goal is to recover  $x \in \mathbb{H}^d$  from the set of projections  $y_n = P_{W_n}(x)$ , with  $n \geq 1$ .

We focus on the following iterative algorithm. Let  $x_0 \in \mathbb{H}^d$  be an arbitrary initial estimate. Produce updated estimates  $x_n \in \mathbb{H}^d$  for  $x \in \mathbb{H}^d$  by running the following iteration for  $n \geq 1$

$$x_n = x_{n-1} + y_n - P_{W_n}(x_{n-1}). \quad (1.7)$$

The algorithm (1.7) dates at least back to [13] in the context of block-iterative methods for linear equations. Note that in contrast to (1.5), each iteration of (1.7) only acts on a single subspace  $W_n$  and a single measurement  $y_n$ , and for this reason, in the setting of fusion frames we shall refer to (1.7) as a *subspace action method*.

Algorithms of the type (1.7) have a long history. Geometrically, (1.7) is an example of a projection onto convex sets (POCS) algorithm, and in particular  $x_n$  is simply the orthogonal projection of  $x_{n-1}$  onto the convex set  $\{u \in \mathbb{H}^d : P_{W_n}(u) = y_n\}$ . The algorithm (1.7) also falls into the class of block iterative methods related to the Kaczmarz algorithm, e.g., [13, 25]. For example, if each  $W_n = \text{span}(\varphi_n)$  is one dimensional, then  $P_{W_n}(x) = \varphi_n \langle x, \varphi_n \rangle / \|\varphi_n\|^2$  and (1.7) reduces to the familiar Kaczmarz algorithm for recovering  $x \in \mathbb{R}^d$  from the linear measurements  $y_n = \langle x, \varphi_n \rangle, n \geq 1$ ,

$$x_n = x_{n-1} + \frac{y_n - \langle x_{n-1}, \varphi_n \rangle}{\|\varphi_n\|^2} \varphi_n, \quad (1.8)$$

for example, see [21, 30]. Randomized versions of this algorithm were recently studied for two-dimensional subspaces in [26] and for general subspaces in [25].

It is useful to note differences between the fusion frame algorithm (1.5) and the subspace action method (1.7). The algorithm (1.5) requires access to the entire fusion frame system  $\{(W_n, v_n)\}_{n \in I}$  and the full set of projections  $\{y_n\}_{n \in I}$  at each iteration and also requires some knowledge of the fusion frame bounds  $A, B$ . For high dimensional problems memory issues might pose challenges to storing or using the entire fusion frame system at once. Moreover, in practice, one might receive access to the fusion frame measurements  $y_n = P_{W_n}(x)$  in a streaming manner for which an online algorithm such as (1.7) might be suitable.

Unfortunately, unlike (1.5), the algorithm (1.7) is sensitive to the order in which it receives the inputs  $y_n = P_{W_n}(x)$ , and an inappropriate ordering of the subspaces can lead to poor performance. Recent work on the Kaczmarz algorithm in [30] indicates that a randomized selection of  $y_n = P_{W_n}(x)$  can circumvent such ordering issues and leads to fast convergence. In view of this, our analysis of (1.7) will mainly focus on the case where the subspaces  $\{W_n\}_{n=1}^\infty$  in  $\mathbb{H}^d$  are independent identically distributed (i.i.d.) versions of a random  $k$ -dimensional subspace  $W \subset \mathbb{H}^d$  (this will include the special case of deterministic fusion frames that have been suitably randomized).

**1.2. Overview and main results.** The main goal of this work is to provide error bounds for the subspace action algorithm (1.7) when it is driven by random subspaces  $\{W_n\}_{n=1}^\infty$ , and to determine which choices of randomization lead to fast convergence in (1.7). The main contributions of this work can be summarized as follows:

- (1) As necessary background lemmas, the error bounds of [11] are extended to the setting of fusion frames and, in particular, they provide bounds on error moments and almost sure convergence rates for the algorithm (1.7).

- (2) We address the following main question. For which choices of i.i.d. random  $k$ -dimensional subspaces  $\{W_n\}_{n=1}^\infty$  of  $\mathbb{H}^d$  does the algorithm (1.7) converge quickly? Specifically, we describe random subspaces with minimal Kaczmarz bounds (the Kaczmarz bounds in turn provide upper bounds for the algorithm's error moments). Uniqueness of minimizers is discussed in special cases.

The paper is organized as follows. In Section 2 we provide necessary background on random subspaces and Kaczmarz bounds. Section 3 provides basic error bounds for the subspace action algorithm (1.7) when the subspaces  $\{W_n\}_{n=1}^\infty$  are i.i.d. versions of a random subspace  $W$ ; these results are immediate generalizations of the error bounds in [11] and give upper bounds on error moments of (1.7) in terms of Kaczmarz bounds of  $W$ .

In Section 4 we address which distributions on a  $k$ -dimensional random subspace  $W$  have minimal Kaczmarz bounds. These distributions in turn lead to the smallest upper bounds on the error moments obtained in Section 3. Our first main results, Theorem 4.6 and Corollary 4.11, show that a distribution achieves the minimal Kaczmarz bound precisely when its Kaczmarz bound is tight. In particular, the invariant measure on the Grassmannian  $G(k, d)$  is a minimizer for both the Kaczmarz bound of order  $s$  and the logarithmic Kaczmarz bound. In Section 5 we address uniqueness of the minimizing distributions from Section 4; Theorem 5.1 shows that the invariant measure on  $G(1, d)$  uniquely minimizes both the logarithmic Kaczmarz bound and the Kaczmarz bound of order  $s$  when  $0 < s < 1$  (but it is not generally unique when  $s = 1$ ). Section 6 provides numerical experiments and examples to illustrate our results.

## 2. RANDOM SUBSPACES AND KACZMARZ BOUNDS

We shall primarily be interested in the performance of the algorithm (1.7) when the spaces  $\{W_n\}_{n=1}^\infty \subset \mathbb{H}^d$  are randomly chosen (or are obtained by randomizing a deterministic fusion frame). In this section, we provide some necessary background and notation related to random subspaces.

The Grassmannian  $G(k, d) = G(k, \mathbb{H}^d)$  is the set of all  $k$ -dimensional subspaces of  $\mathbb{H}^d$ . Let  $\mathcal{G} = \cup_{k=0}^d G(k, d)$  denote the collection of all subspaces of  $\mathbb{H}^d$ , let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of  $\mathcal{G}$ , and let  $P$  be a probability measure on  $\mathcal{A}$ . Let  $\mathbb{S}^{d-1} = \mathbb{S}_{\mathbb{H}}^{d-1} = \{x \in \mathbb{H}^d : \|x\| = 1\}$  denote the unit-sphere in  $\mathbb{H}^d$ . To begin, we simply assume  $W$  is a random subspace defined on the probability space  $(\mathcal{G}, \mathcal{A}, P)$ . The error bounds in Section 3 require the following quantitative notion of how nondegenerate a random subspace is.

**Definition 2.1.** Given  $s > 0$ , the *Kaczmarz bound*  $0 \leq \alpha_s \leq 1$  of order  $s$  for a random subspace  $W \subset \mathbb{H}^d$  is defined as

$$\alpha_s = \sup_{x \in \mathbb{S}^{d-1}} \left( \mathbb{E} \left[ (1 - \|P_W(x)\|^2)^s \right] \right)^{1/s}.$$

The *logarithmic Kaczmarz bound*  $0 \leq \alpha_{\log} \leq 1$  for a random subspace  $W \subset \mathbb{H}^d$  is defined as

$$\alpha_{\log} = \sup_{x \in \mathbb{S}^{d-1}} \exp \left( \mathbb{E} \left[ \log (1 - \|P_W(x)\|^2) \right] \right).$$

In particular,

$$\forall x \in \mathbb{S}^{d-1}, \quad \left( \mathbb{E} \left[ (1 - \|P_W(x)\|^2)^s \right] \right)^{1/s} \leq \alpha_s, \quad (2.1)$$

and

$$\forall x \in \mathbb{S}^{d-1}, \quad \exp \left( \mathbb{E} \left[ \log \left( 1 - \|P_W(x)\|^2 \right) \right] \right) \leq \alpha_{\log}. \quad (2.2)$$

We shall say that the Kacmarz bound or logarithmic Kacmarz bound is *tight* if equality holds in (2.1) or (2.2) respectively.

See [11] for further discussion of Kacmarz bounds. The motivation for the logarithmic Kacmarz bound (2.2) comes from considering the limit as  $s \rightarrow 0$  in (2.1), see Lemma 2.2.

**Lemma 2.2.** *Let  $X$  be a random variable. If  $s_1 \geq s_2 > 0$  then*

$$(\mathbb{E}|X|^{s_2})^{1/s_2} \leq (\mathbb{E}|X|^{s_1})^{1/s_1}. \quad (2.3)$$

*Moreover, if  $\mathbb{E}|X|^s < \infty$  for some  $s > 0$  then*

$$\inf_{s>0} (\mathbb{E}|X|^s)^{1/s} = \lim_{s \rightarrow 0} (\mathbb{E}|X|^s)^{1/s} = \exp(\mathbb{E} \log |X|). \quad (2.4)$$

The inequality (2.3) is known as *Lyapunov's inequality*, see page 193 in [29], and the limit (2.4) can, for example, be found on page 71 in [28].

Although we shall mainly analyze the algorithm (1.7) when it is driven by random subspaces, it is useful to note that this also includes deterministic fusion frames that have been randomized.

**Example 2.3.** Suppose that  $\{(W_n, v_n)\}_{n=1}^N$  is a fusion frame for  $\mathbb{H}^d$  with frame bounds  $0 < A \leq B < \infty$ . Let  $W$  be the random subspace defined by

$$\forall 1 \leq k \leq N, \quad \Pr[W = W_k] = \frac{v_k^2}{\sum_{n=1}^N v_n^2}. \quad (2.5)$$

Then  $W$  has a Kacmarz bound  $\alpha_1$  of order  $s = 1$  which satisfies

$$\alpha_1 \leq 1 - \frac{A}{\sum_{n=1}^N v_n^2} < 1. \quad (2.6)$$

In the special case when  $\{(W_n, v_n)\}_{n=1}^N$  is a tight fusion frame then the fusion frame bounds are given by  $A = B = \frac{1}{d} \sum_{n=1}^N v_n^2 \dim(W_n)$ , e.g., see Chapter 13 in [9], and in this case the random subspace  $W$  has a tight Kacmarz bound of order  $s = 1$  given by

$$\alpha_1 = 1 - \frac{\sum_{n=1}^N v_n^2 \dim(W_n)}{d \sum_{n=1}^N v_n^2}.$$

**Example 2.4.** Let  $\{e_n\}_{n=1}^d$  be an orthonormal basis for  $\mathbb{H}^d$ , and let  $W$  be the random 1-dimensional subspace defined by

$$\forall 1 \leq n \leq d, \quad \Pr[W = \text{span}(e_n)] = 1/d. \quad (2.7)$$

Then

$$(\mathbb{E} (1 - \|P_W(x)\|^2)^s)^{1/s} = \left( \frac{1}{d} \sum_{n=1}^d (1 - |\langle x, e_n \rangle|^2)^s \right)^{1/s}. \quad (2.8)$$

When  $0 < s < 1$ , it can be verified that that supremum of (2.8) over all  $x \in \mathbb{S}^{d-1}$  equals  $(1 - 1/d)$  and is, for example, attained by  $x = d^{-1/2} \sum_{n=1}^d e_n$ . So, when  $0 < s < 1$ ,  $W$  has the Kacmarz bound of order  $s$  given by  $\alpha_s = (1 - 1/d)$ .

When  $1 \leq s < \infty$ , it can be verified that that supremum of (2.8) over all  $x \in \mathbb{S}^{d-1}$  equals  $(1 - 1/d)^{1/s}$  and is, for example, attained by  $x = e_1$ . So, when  $1 \leq s < \infty$ ,  $W$  has the Kaczmarz bound of order  $s$  given by  $\alpha_s = (1 - 1/d)^{1/s}$ .

### 3. ERROR BOUNDS

This section states error bounds for the subspace action algorithm (1.7). The results of this section are directly motivated by analogous results in [11] for the standard Kaczmarz algorithm. We omit most proofs in this section since they are almost identical to their Kaczmarz counterparts in [11].

The following basic error bound is proven in the same manner as Proposition 3.1 in [11]. Notice that (1.7) implies  $x - x_n = P_{W_n^\perp}(x - x_{n-1})$ , hence (3.1).

**Lemma 3.1.** *The error in the algorithm (1.7) satisfies*

$$(x - x_n) = P_{W_n^\perp} \cdots P_{W_2^\perp} P_{W_1^\perp}(x - x_0) \quad (3.1)$$

and

$$\|x - x_n\|^2 = \|x - x_{n-1}\|^2 - \|P_{W_n}(x - x_{n-1})\|^2, \quad (3.2)$$

and

$$\|x - x_n\|^2 = \|x - x_0\|^2 \prod_{k=1}^n \left( 1 - \left\| P_{W_k} \left( \frac{x - x_{k-1}}{\|x - x_{k-1}\|} \right) \right\|^2 \right). \quad (3.3)$$

The next theorem is proven in the same manner as Theorem 4.1 in [11].

**Theorem 3.2.** *Let  $\{W_n\}_{n=1}^\infty$  be independent random subspaces of  $\mathbb{H}^d$ . Assume that each  $W_n$  has the common Kaczmarz bound  $0 \leq \alpha_s \leq 1$  of order  $s$ . The error of the subspace action algorithm (1.7) satisfies*

$$(\mathbb{E}\|x - x_n\|^{2s})^{1/s} \leq \alpha_s^n \|x - x_0\|^2. \quad (3.4)$$

Moreover, if the common Kaczmarz bound is tight, then equality holds in (3.4).

The next result follows from Theorem 3.2, Lemma 2.2 and Definition 2.1. The proof is similar to the proof of Theorem 6.3 in [11], but we include details for the sake of completeness.

**Theorem 3.3.** *Suppose that the random subspaces  $\{W_n\}_{n=1}^\infty$  of  $\mathbb{H}^d$  are independent and identically distributed versions of a random subspace  $W$  that has the logarithmic Kaczmarz bound  $0 \leq \alpha_{\log} \leq 1$ . The error of the subspace action algorithm (1.7) satisfies*

$$\exp(\mathbb{E}[\log \|x - x_n\|^2]) \leq \alpha_{\log}^n \|x - x_0\|^2. \quad (3.5)$$

*Proof.* Let  $\epsilon > 0$  be arbitrary. Using Lemma 2.2, for every  $x \in \mathbb{S}^{d-1}$  there exists  $s_x > 0$  such that

$$(\mathbb{E}(1 - \|P_W(x)\|^2)^{s_x})^{1/s_x} \leq \alpha_{\log} + \epsilon. \quad (3.6)$$

By the Lebesgue dominated convergence theorem, for every  $x \in \mathbb{S}^{d-1}$

$$\lim_{\|y\|=1, y \rightarrow x} (\mathbb{E}(1 - \|P_W(y)\|^2)^{s_x})^{1/s_x} = (\mathbb{E}(1 - \|P_W(x)\|^2)^{s_x})^{1/s_x} \leq \alpha_{\log} + \epsilon. \quad (3.7)$$

Thus for every  $x \in \mathbb{S}^{d-1}$  there exists an open neighborhood  $U_x \subset \mathbb{S}^{d-1}$  about  $x$  such that

$$\forall y \in U_x, \quad (\mathbb{E}(1 - \|P_W(y)\|^2)^{s_x})^{1/s_x} \leq \alpha_{\log} + 2\epsilon. \quad (3.8)$$

By compactness there exists a finite set  $\{x_n\}_{n=1}^N \subset \mathbb{S}^{d-1}$  such that  $\cup_{n=1}^N U_{x_n}$  covers  $\mathbb{S}^{d-1}$ . Letting  $s_\epsilon^* = \min\{s_{x_n}\}_{n=1}^N > 0$  and using (2.3) gives

$$\forall x \in \mathbb{S}^{d-1}, \quad (\mathbb{E}(1 - \|P_W(x)\|^2)^{s_\epsilon^*})^{1/s_\epsilon^*} \leq \alpha_{\log} + 2\epsilon. \quad (3.9)$$

Thus by Lemma 2.2 and Theorem 3.2

$$\exp(\mathbb{E} \log \|x - x_n\|^2) \leq (\mathbb{E} \|x - x_n\|^{2s_\epsilon^*})^{1/s_\epsilon^*} \leq (\alpha_{\log} + 2\epsilon)^n \|x - x_0\|^2. \quad (3.10)$$

Since  $\epsilon > 0$  is arbitrary, (3.10) yields (3.5) as required.  $\square$

While the error bounds (3.4) and (3.5) are natural, it is important to note that they are simply upper bounds and need not be sharp. The following example shows that in certain cases the error moments can be much smaller than the estimates obtained using Kaczmarz bounds and thereby illustrates some practical limitations of Theorem 3.2.

**Example 3.4.** Let  $\{e_k\}_{k=1}^d$  be an orthonormal basis for  $\mathbb{H}^d$  and let  $V_k = \text{span}(e_k)$ . Let  $W$  be the random 1-dimensional subspace defined by

$$\forall 1 \leq k \leq d, \quad \Pr[W = V_k] = 1/d.$$

Suppose that  $\{W_n\}_{n=1}^N$  are i.i.d. versions of  $W$ .

To begin, recall that when  $0 < s < 1$ ,  $W$  has the Kaczmarz bound  $\alpha_s = (1 - 1/d)$ , see Example 2.4. So, by Theorem 3.2 the error for subspace action algorithm (1.7) satisfies

$$\mathbb{E} \|x - x_N\|^{2s} \leq (1 - 1/d)^{sN} \|x - x_0\|^{2s}. \quad (3.11)$$

Next, a more detailed analysis will show that the error bound (3.11) can be significantly improved in this example. Let the random variable  $K_j$  denote the number of  $\{W_n\}_{n=1}^N$  which equal  $V_j$ . Since the  $\{e_n\}_{n=1}^d$  are orthonormal, the projections  $P_{W_n^\perp}$  commute, namely  $P_{W_j^\perp} P_{W_k^\perp} = P_{W_k^\perp} P_{W_j^\perp}$  for all  $j, k$ . This together with (3.1) gives

$$\|x - x_N\|^2 = \|P_{W_N^\perp} \cdots P_{W_1^\perp} (x_0 - x)\|^2 = \|P_{V_1^\perp}^{K_1} P_{V_2^\perp}^{K_2} \cdots P_{V_d^\perp}^{K_d} (x - x_0)\|^2.$$

Since the  $\{e_j\}_{j=1}^d$  are orthonormal it can be verified that if  $K_j \neq 0$  holds for all  $1 \leq j \leq d$ , then  $P_{V_1^\perp}^{K_1} P_{V_2^\perp}^{K_2} \cdots P_{V_d^\perp}^{K_d} = 0$ . Let  $A_j$  denote the event that  $K_j = 0$ , and note that  $\Pr[A_1 \cap A_2 \cap \cdots \cap A_n] = (1 - n/d)^N$ . Let  $A = \bigcup_{j=1}^d A_j$ , and let  $\chi_A$  denote the indicator function of the event  $A$ . Thus

$$\mathbb{E} \|x - x_N\|^{2s} = \mathbb{E} \left( \|P_{V_1^\perp}^{K_1} P_{V_2^\perp}^{K_2} \cdots P_{V_d^\perp}^{K_d} (x - x_0)\|^{2s} \chi_A \right). \quad (3.12)$$

To obtain an upper bound on  $\mathbb{E}\|x - x_N\|^{2s}$ , note that the projections  $P_{V_j^\perp}$  have norm one, and apply the inclusion-exclusion principle to (3.12) as follows

$$\begin{aligned} \mathbb{E}\|x - x_N\|^{2s} &\leq \|x - x_0\|^{2s} \mathbb{E}[\chi_A] = \|x - x_0\|^{2s} \Pr\left(\bigcup_{j=1}^d A_j\right) \\ &= \|x - x_0\|^{2s} \sum_{k=1}^d (-1)^{k+1} \binom{d}{k} \Pr\left(\bigcap_{j=1}^k A_j\right) \\ &= \|x - x_0\|^{2s} \sum_{k=1}^d (-1)^{k+1} \binom{d}{k} \left(1 - \frac{k}{d}\right)^N. \end{aligned} \quad (3.13)$$

Keeping only the  $k = 1$  term in the sum (3.13) gives

$$\mathbb{E}\|x_N - x\|^{2s} \leq d(1 - 1/d)^N \|x - x_0\|^{2s}. \quad (3.14)$$

When  $0 < s < 1$  (and especially when  $s$  is near 0), the error bound (3.14) is smaller than the error bound (3.11) for adequately large  $N$ .

Finally, it is worth noting that the upper bound (3.13) cannot be significantly improved. To see this, consider the case when  $x - x_0$  satisfies  $|\langle (x - x_0), e_n \rangle| \geq C > 0$  for all  $1 \leq n \leq d$ . Since the  $\{e_n\}_{n=1}^d$  are orthonormal, it can be shown that if  $K_j = 0$  for some  $1 \leq j \leq n$  then

$$\|P_{V_1^\perp}^{K_1} P_{V_2^\perp}^{K_2} \cdots P_{V_d^\perp}^{K_d}(x - x_0)\| \geq C.$$

Thus (3.12) and similar steps as for (3.13) give

$$\mathbb{E}\|x_n - x\|^{2s} \geq C^{2s} \mathbb{E}(\chi_A) = C^{2s} \Pr\left(\bigcup_{j=1}^d A_j\right) = C^{2s} \sum_{k=1}^d (-1)^{k+1} \binom{d}{k} \left(1 - \frac{k}{d}\right)^N. \quad (3.15)$$

Theorems 3.2 and 3.3 give upper bounds on error moments for the algorithm (1.7) in terms of the Kaczmarz bounds  $\alpha_s$  and  $\alpha_{\log}$ . Kaczmarz bounds can similarly be used to control rates of almost sure convergence. The next theorem can be proven in the same manner as either one of the two different proofs of Theorem 6.2 in [11].

**Theorem 3.5.** *Let  $\{W_k\}_{k=1}^\infty$  be independent random subspaces of  $\mathbb{H}^d$ . Let  $s > 0$  be fixed and suppose that each  $W_k$  has the common Kaczmarz bound  $0 < \alpha_s < 1$  of order  $s$ . The error in the subspace action algorithm (1.7) satisfies*

$$\forall 0 < r < 1/\alpha_s, \quad \lim_{n \rightarrow \infty} r^n \|x - x_n\|^2 = 0, \quad \text{almost surely.}$$

The next theorem is proven in the same manner as Theorem 6.3 in [11].

**Theorem 3.6.** *Suppose that the random subspaces  $\{W_n\}_{n=1}^\infty$  of  $\mathbb{H}^d$  are independent and identically distributed versions of a random subspace  $W$  that has the logarithmic Kaczmarz bound  $0 < \alpha_{\log} < 1$ . The error in the subspace action algorithm (1.7) satisfies*

$$\forall 0 < r < 1/\alpha_{\log}, \quad \lim_{n \rightarrow \infty} r^n \|x - x_n\|^2 = 0, \quad \text{almost surely.}$$



Note that, similar to [30], the error bounds of this section apply to deterministic fusion frames that have been randomized, for example as in Example 2.3. In particular, if each step of the algorithm (1.7) is run using a random instance of  $W$  generated by (2.5), then the error bounds of this section apply, for example with  $\alpha_1$  given by (2.6).

#### 4. MINIMAL KACZMARZ BOUNDS AND OPTIMAL DISTRIBUTIONS

Theorems 3.2 and 3.3 show that smaller Kaczmarz bounds  $\alpha_s$  or  $\alpha_{\log}$  yield smaller upper bounds on various error moments in (3.4) and (3.5). In view of this, it is natural to ask which distributions on the probabilistic fusion frame  $W$  give the smallest Kaczmarz bound  $\alpha_s$  or smallest logarithmic Kaczmarz bound  $\alpha_{\log}$ . For this question to be nontrivial, we shall restrict our attention to random subspaces  $W \in G(k, d)$  with fixed dimension  $1 \leq k < d$ . Indeed, without fixing the dimension  $k$ , the trivial case  $W = \mathbb{H}^d$  would ensure that  $\alpha_s = \alpha_{\log} = 0$  and that the algorithm (1.7) converges exactly after one step.

So, in this section we shall assume that  $W \in G(k, d)$  is a random subspace, and we consider the question of which distributions on  $W$  minimize

$$\sup_{x \in \mathbb{S}^{d-1}} (\mathbb{E}(1 - \|P_W(x)\|^2)^s)^{1/s} \quad \text{or} \quad \sup_{x \in \mathbb{S}^{d-1}} \exp(\mathbb{E}[\log(1 - \|P_W(x)\|^2)]).$$

Equivalently, we seek to determine which Borel probability measures  $\mu$  on  $G(k, d)$  minimize each of the quantities

$$F_{\alpha_s}(\mu) = \sup_{x \in \mathbb{S}^{d-1}} \int_{G(k, d)} (1 - \|P_W(x)\|^2)^s d\mu(W), \quad (4.1)$$

$$F_{\alpha_{\log}}(\mu) = \sup_{x \in \mathbb{S}^{d-1}} \int_{G(k, d)} \log(1 - \|P_W(x)\|^2) d\mu(W). \quad (4.2)$$

For related potential theoretic problems involving frames and fusion frames see [6, 12, 23].

It will be convenient to address (4.1) and (4.2) as special cases of the more general problem of finding which Borel probability measures  $\mu$  on  $X$  minimize

$$\sup_{x \in X} \int_Y K(x, y) d\mu(y), \quad (4.3)$$

where  $X, Y$  are suitable homogeneous spaces and  $K : X \times Y \rightarrow \mathbb{R} \cup \{-\infty\}$  is a suitable Borel measurable kernel.

**4.1. Kernels and potentials on homogeneous spaces.** We shall assume throughout this section that  $X$  and  $Y$  are homogeneous spaces for a group  $G$ . Specifically, we assume that  $(X, d_X)$  and  $(Y, d_Y)$  are compact metric spaces and that  $G$  is a compact topological group that acts isometrically and transitively on  $X$  and  $Y$ . Recall that  $G$  acts isometrically on  $X$  if

$$\forall g \in G, \forall x_1, x_2 \in X, \quad d_X(gx_1, gx_2) = d_X(x_1, x_2),$$

and  $G$  acts transitively on  $X$  if

$$\forall x_1, x_2 \in X, \exists g \in G, \quad \text{such that } x_1 = gx_2.$$

In a given compact topological space  $X$ , we let  $\mathcal{B}(X)$  denote the Borel  $\sigma$ -algebra and let  $\mathcal{M}(X)$  denote the set of all Borel probability measures on  $(X, \mathcal{B}(X))$ .

By the construction of Haar measure, see Part I, Section 1 in [24], there exist unique Radon probability measures  $m_X \in \mathcal{M}(X)$  and  $m_Y \in \mathcal{M}(Y)$  with the  $G$ -invariance property that for all  $B_1 \in \mathcal{B}(X)$  and  $B_2 \in \mathcal{B}(Y)$

$$\forall g \in G, \quad m_X(g(B_1)) = m_X(B_1) \quad \text{and} \quad m_Y(g(B_2)) = m_Y(B_2), \quad (4.4)$$

where  $g(B_i) = \{g(b) : b \in B_i\}$ . The invariant measures  $m_X$  and  $m_Y$  are *strictly positive*, i.e.,  $m_X(\mathcal{O}_1) > 0$  and  $m_Y(\mathcal{O}_2) > 0$  holds for all nonempty open sets  $\mathcal{O}_1 \subset X$  and  $\mathcal{O}_2 \subset Y$ . For compact spaces, this is a consequence of the transitivity of the group action.

In metric spaces, recall that a function  $f : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is *upper semi-continuous* if  $\limsup_{n \rightarrow \infty} f(x_n) \leq f(x)$  whenever  $\lim_{n \rightarrow \infty} x_n = x$  and  $x \in X$ ,  $\{x_n\}_{n=1}^\infty \subset X$ . In particular, if  $f(x_0) < c$  for some  $x_0 \in X$ ,  $c \in \mathbb{R}$  then for every  $\epsilon > 0$  there exists an open neighborhood  $\mathcal{O} \subset X$  containing  $x_0$  such that if  $x \in \mathcal{O}$  then  $f(x) < c + \epsilon$ .

**Definition 4.1.** A Borel measurable function  $K : X \times Y \rightarrow \mathbb{R} \cup \{-\infty\}$  will be said to be an *admissible kernel* if the following four conditions hold

$$\exists B_K \in [0, \infty), \forall x \in X, \forall y \in Y, \quad -\infty \leq K(x, y) \leq B_K, \quad (4.5)$$

and

$$\forall y \in Y, \quad \int_X |K(x, y)| dm_X(x) < \infty, \quad (4.6)$$

and

$$\forall y \in Y, \text{ the function } K(\cdot, y) \text{ is upper semi-continuous,} \quad (4.7)$$

and

$$\forall g \in G, \forall x \in X, \forall y \in Y, \quad K(g(x), y) = K(x, g^{-1}(y)). \quad (4.8)$$

**Lemma 4.2.** If  $K$  is an admissible kernel then there exists a constant  $C_K \in \mathbb{R}$  such that

$$\forall y \in Y, \quad \int_X K(x, y) dm_X(x) = C_K. \quad (4.9)$$

In view of (4.9), we shall say that an admissible kernel  $K$  has constant  $C_K$ .

*Proof.* The  $G$ -invariance of  $m_X$  and (4.8) imply that for any  $g \in G$  and any  $y \in Y$

$$\int_X K(x, g(y)) dm_X(x) = \int_X K(g^{-1}(x), y) dm_X(x) = \int_X K(x, y) dm_X(x).$$

Since the action of  $g$  on  $Y$  is transitive and by (4.6) this completes the proof.  $\square$

**Definition 4.3.** If  $K$  is an admissible kernel and  $\mu \in \mathcal{M}(Y)$  then the associated *potential function*  $U_K^\mu : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is defined by

$$\forall x \in X, \quad U_K^\mu(x) = \int_Y K(x, y) d\mu(y).$$

**Lemma 4.4.** Let  $K$  be an admissible kernel with constant  $C_K$ . Given  $\mu \in \mathcal{M}(Y)$ , suppose that  $U_K^\mu(x_1) = U_K^\mu(x_2)$  for all  $x_1, x_2 \in X$ . Then  $U_K^\mu(x) = C_K$  for all  $x \in X$ .

*Proof.* Integrating both sides of (4.9) with respect to  $d\mu$  and using (4.5) to apply the Fubini-Tonelli theorem gives that for any  $x_0 \in \mathbb{S}^{d-1}$

$$\begin{aligned} C_K &= \int_Y \int_X K(x, y) dm_X(x) d\mu(y) = \int_X \int_Y K(x, y) d\mu(y) dm_X(x) \\ &= \int_X U_K^\mu(x) dm_X(x) = \int_X U_K^\mu(x_0) dm_X(x) = U_K^\mu(x_0). \end{aligned}$$

□

**Lemma 4.5.** *If  $K$  is an admissible kernel and  $\mu \in \mathcal{M}(Y)$  then the potential function  $U_K^\mu$  is upper semi-continuous.*

*Proof.* Let  $x \in X$  be arbitrary and suppose that  $\{x_n\}_{n=1}^\infty \subset X$  satisfies  $\lim_{n \rightarrow \infty} x_n = x$ . Letting  $B_K$  be as in (4.5), it follows from (4.7) that for all  $y \in Y$

$$B_K - K(x, y) \leq B_K - \limsup_{n \rightarrow \infty} K(x_n, y) = \liminf_{n \rightarrow \infty} (B_K - K(x_n, y)). \quad (4.10)$$

Applying Fatou's lemma to the nonnegative functions  $g_n(y) = B_K - K(x_n, y) \geq 0$  and using (4.10) gives

$$\begin{aligned} B_K - U_K^\mu(x) &= \int_Y (B_K - K(x, y)) d\mu(y) \\ &\leq \int_Y \liminf_{n \rightarrow \infty} (B_K - K(x_n, y)) d\mu(y) \\ &= \int_Y \liminf_{n \rightarrow \infty} g_n(y) d\mu(y) \\ &\leq \liminf_{n \rightarrow \infty} \int_Y g_n(y) d\mu(y) \\ &= B_K - \limsup_{n \rightarrow \infty} \int_Y K(x_n, y) d\mu(y) \\ &= B_K - \limsup_{n \rightarrow \infty} U_K^\mu(x_n). \end{aligned}$$

This shows that  $U_K^\mu$  is upper semi-continuous. □

The following theorem characterizes minimizers  $\mu \in \mathcal{M}(Y)$  of (4.3) in terms of the potential function  $U_K^\mu$  being constant.

**Theorem 4.6.** *Let  $K$  be an admissible kernel with constant  $C_K$  and let  $\mu_0 \in \mathcal{M}(Y)$ . The following are equivalent:*

- (1) *The function  $U_K^{\mu_0}$  is constant,*
- (2) *For every  $x \in X$ , there holds  $U_K^{\mu_0}(x) = C_K$ ,*
- (3)  *$\mu_0$  minimizes (4.3), that is,  $\inf_{\mu \in \mathcal{M}(Y)} \sup_{x \in X} U_K^\mu(x) = \sup_{x \in X} U_K^{\mu_0}(x)$ .*

*Proof.* (1)  $\iff$  (2). Lemma 4.4 shows the equivalence of (1) and (2).

(2)  $\implies$  (3). Proceed by contradiction and assume that there exists  $\nu \in \mathcal{M}(Y)$  such that

$$\sup_{x \in X} U_K^{\mu_0}(x) > \sup_{x \in X} U_K^\nu(x). \quad (4.11)$$

The definitions of  $U_K^\nu$  and  $C_K$ , along with (4.11), yield

$$\begin{aligned} C_K &= \sup_{x \in X} U_K^{\mu_0}(x) > \sup_{x \in X} U_K^\nu(x) \geq \int_X U_K^\nu(x) dm_X(x) \\ &= \int_X \int_Y K(x, y) d\nu(y) dm_X(x) \\ &= \int_Y \int_X K(x, y) dm_X(x) d\nu(y) \\ &= \int_Y C_K d\nu(y) = C_K. \end{aligned}$$

Thus,  $C_K > C_K$  gives the desired contradiction.

(3)  $\implies$  (1). Property (4.8) and the  $G$ -invariance of  $m_Y$  imply that for any  $x \in X$  and any  $g \in G$

$$\begin{aligned} U_K^{m_Y}(g(x)) &= \int_Y K(g(x), y) dm_Y(y) = \int_Y K(x, g^{-1}(y)) dm_Y(y) \\ &= \int_Y K(x, y) dm_Y(y) = U_K^{m_Y}(x). \end{aligned} \quad (4.12)$$

Since the action of  $G$  on  $X$  is transitive, (4.12) means that  $U_K^{m_Y}(x_1) = U_K^{m_Y}(x_2)$  for all  $x_1, x_2 \in X$ . So, Lemma 4.4 shows that  $U_K^{m_Y}(x) = C_K$  for all  $x \in X$ . The previously proven implication (2)  $\implies$  (3) shows that

$$\inf_{\mu \in \mathcal{M}(Y)} \sup_{x \in X} U_K^\mu(x) = \sup_{x \in X} U_K^{m_Y}(x) = C_K. \quad (4.13)$$

The assumption (3) together with (4.13) gives

$$\sup_{x \in X} U_K^{\mu_0}(x) = \inf_{\mu \in \mathcal{M}(Y)} \sup_{x \in X} U_K^\mu(x) = \sup_{x \in X} U_K^{m_Y}(x) = C_K.$$

We proceed by contradiction and assume that  $U_K^{\mu_0}$  is not constant, so that there exists  $x_0 \in X$  and  $\epsilon > 0$  with  $U_K^{\mu_0}(x_0) < C_K - 2\epsilon$ . Since  $U_K^{\mu_0}(x)$  is upper semi-continuous, there exists an open neighborhood  $B \subset X$  containing  $x_0$  such that

$$\forall x \in B, \quad U_K^{\mu_0}(x) < C_K - \epsilon.$$

Since  $m_X$  is strictly positive, we have  $m_X(B) > 0$ . Thus,

$$\begin{aligned} \int_X U_K^{\mu_0}(x) dm_X(x) &= \int_B U_K^{\mu_0}(x) dm_X(x) + \int_{X \setminus B} U_K^{\mu_0}(x) dm_X(x) \\ &\leq (C_K - \epsilon) m_X(B) + C_K m_X(X \setminus B) \\ &= C_K - \epsilon m_X(B) < C_K. \end{aligned} \quad (4.14)$$

On the other hand, the definitions of  $U_K^{\mu_0}$  and  $C_K$  along with the Fubini-Tonelli theorem give

$$\begin{aligned} \int_X U_K^{\mu_0}(x) dm_X(x) &= \int_X \int_Y K(x, y) d\mu_0(y) dm_X(x) \\ &= \int_Y \int_X K(x, y) dm_X(x) d\mu_0(y) \\ &= \int_Y C_K d\mu_0(y) = C_K. \end{aligned} \quad (4.15)$$

Equations (4.14) and (4.15) yield the desired contradiction  $C_K < C_K$ .  $\square$

Note that the assumption of upper semi-continuity was only needed for the proof of the implication (3)  $\implies$  (1). Also, since (4.12) and the subsequent remarks show that  $U_K^{m_Y}(x) = C_K$  holds for all  $x \in X$ , we have the following corollary.

**Corollary 4.7.** *The invariant measure  $m_Y \in \mathcal{M}(Y)$  satisfies*

$$\inf_{\mu \in \mathcal{M}(Y)} \sup_{x \in X} U_K^\mu(x) = \sup_{x \in X} U_K^{m_Y}(x).$$

*In other words, the invariant measure  $\mu = m_Y$  is a minimizer of (4.3).*

**4.2. Optimal distributions for subspace actions.** We now specialize Theorem 4.6 to the problems (4.1) and (4.2) of determining which distributions give minimal Kaczmarz bounds for the algorithm (1.7). We consider the case when  $X = \mathbb{S}^{d-1}$ ,  $Y = G(k, d)$ , and  $G$  is either the orthogonal group  $\mathcal{O}(d)$  or unitary group  $\mathcal{U}(d)$  depending on whether  $\mathbb{H} = \mathbb{R}$  or  $\mathbb{C}$ .

If  $X = \mathbb{S}^{d-1}$  is endowed with the norm metric  $d_X(u, v) = \|u - v\|$  from  $\mathbb{H}^d$ , then  $G$  acts isometrically on  $X$ , and we let  $\sigma_{d-1} = m_X$  denote the invariant measure as in (4.4). Next, endow  $Y = G(k, d)$  with a metric  $d_Y$  as follows. Given subspaces  $V, W \in G(k, d)$ , let  $P_V, P_W$  denote the matrix representations (with respect to the canonical basis) of the orthogonal projections onto  $V$  and  $W$  respectively, and define the metric  $d_Y(V, W) = \|P_V - P_W\|_{\text{Frob}}$  on  $G(k, d)$ . The group  $G$  acts isometrically on  $Y$  with this metric, and we let  $\Gamma_{k,d} = m_Y \in \mathcal{M}(G(k, d))$  denote the invariant measure as in (4.4). In particular, the invariant measure  $\Gamma_{k,d}$  satisfies

$$\forall S \in \mathcal{B}(G(k, d)), \quad \forall g \in G \quad \Gamma_{k,d}(g(S)) = \Gamma_{k,d}(S),$$

where  $g(S) = \{g(W) : W \in S\}$ .

The following examples show that the kernels associated to problems (4.1) and (4.2) are admissible as in Definition 4.1, and hence can be treated using the results of Section 4.1.

**Example 4.8.** Fix  $s > 0$  and define  $K_{\alpha_s} : \mathbb{S}^{d-1} \times G(k, d) \rightarrow \mathbb{R} \cup \{-\infty\}$  by

$$K_{\alpha_s}(x, W) = (1 - \|P_W(x)\|^2)^s. \quad (4.16)$$

Then  $K_{\alpha_s}$  is an admissible kernel. Conditions (4.5), (4.7), and (4.8) are easy to verify, and (4.6) holds since  $0 \leq K_{\alpha_s}(x, W) \leq 1$  holds for all  $x \in \mathbb{S}^{d-1}$  and  $W \in G(k, d)$ .

**Example 4.9.** Define  $K_{\alpha_{\log}} : \mathbb{S}^{d-1} \times G(k, d) \rightarrow \mathbb{R} \cup \{-\infty\}$  by

$$K_{\alpha_{\log}}(x, W) = \log(1 - \|P_W(x)\|^2). \quad (4.17)$$

Then  $K_{\alpha_{\log}}$  is an admissible kernel. Conditions (4.7) and (4.8) are easy to verify, and (4.5) holds since  $-\infty \leq K_{\alpha_{\log}}(x, W) \leq 0$  holds for all  $x \in \mathbb{S}^{d-1}$  and  $W \in G(k, d)$ . The next result, Lemma 4.10, verifies the condition (4.6) by direct computation.

**Lemma 4.10.** *Fix  $1 \leq k < d$  and let  $K_{\alpha_{\log}}$  be as in (4.17). Then  $K_{\alpha_{\log}}$  satisfies (4.6). In other words,*

$$\forall W \in G(k, d), \quad \int_{\mathbb{S}^{d-1}} |\log(1 - \|P_W(x)\|^2)| d\sigma_{d-1}(x) < \infty. \quad (4.18)$$

*Proof.* We begin with the real case  $\mathbb{H}^d = \mathbb{R}^d$ . In this case, note that the measure  $\sigma_{d-1}$  is defined by

$$\forall E \in \mathcal{B}(\mathbb{S}^{d-1}), \quad \sigma_{d-1}(E) = \frac{H_{d-1}(E)}{H_{d-1}(\mathbb{S}^{d-1})}, \quad (4.19)$$

where  $H_{d-1}$  denotes the  $(d-1)$ -dimensional Hausdorff measure.

Fix  $1 \leq k < d$ . Given  $x = (x_1, \dots, x_d) \in \mathbb{H}^d$  denote  $x' = (x_1, \dots, x_k) \in \mathbb{H}^k$  and  $x'' = (x_{k+1}, \dots, x_d) \in \mathbb{H}^{d-k}$ . So that  $x = (x', x'')$  and  $\|x\|^2 = \|x'\|^2 + \|x''\|^2$ . Using  $\mathcal{O}(d)$ -invariance of  $\sigma_{d-1}$ , we may assume without loss of generality that  $W$  is the span of the first  $k$  canonical basis vectors, so that  $P_W(x) = x'$ . When  $x \in \mathbb{S}^{d-1}$  we have  $1 - \|P_W(x)\|^2 = 1 - \|x'\|^2 = \|x''\|^2$ .

It will be useful to recall the following change of variables formula in spherical coordinates, e.g., see Appendix D.2 in [17],

$$\int_{R\mathbb{S}^{d-1}} f(x) dH_{d-1}(x) = \int_{-R}^R \left( \int_{\sqrt{R^2 - x_1^2} \mathbb{S}^{d-2}} f(x_1, x_2, \dots, x_d) dH_{d-2}(x_2, \dots, x_d) \right) \frac{R dx_1}{\sqrt{R^2 - x_1^2}}.$$

A repeated application of this leads to the following

$$\int_{\mathbb{S}^{d-1}} f(x) dH_{d-1}(x) = \int_{\mathbb{B}_{d-k}} \left( \int_{\sqrt{1 - \|x''\|^2} \mathbb{S}^{k-1}} f(x', x'') dH_{k-1}(x') \right) \frac{dH_{d-k}(x'')}{\sqrt{1 - \|x''\|^2}}, \quad (4.20)$$

where  $\mathbb{B}_{d-k} = \{x'' \in \mathbb{H}^{d-k} : \|x''\| \leq 1\}$  is the unit-ball in  $\mathbb{H}^{d-k}$ .

Using (4.19) and (4.20) yields

$$\begin{aligned} H_{d-1}(\mathbb{S}^{d-1}) \int_{\mathbb{S}^{d-1}} \log(1 - \|P_W(x)\|^2) d\sigma_{d-1}(x) &= 2 \int_{\mathbb{S}^{d-1}} \log \|x''\| dH_{d-1}(x) \\ &= 2 \int_{\mathbb{B}_{d-k}} \left( \int_{\sqrt{1 - \|x''\|^2} \mathbb{S}^{k-1}} \log \|x''\| dH_{k-1}(x') \right) \frac{dH_{d-k}(x'')}{\sqrt{1 - \|x''\|^2}} \\ &= 2 \int_{\mathbb{B}_{d-k}} H_{k-1}(\sqrt{1 - \|x''\|^2} \mathbb{S}^{k-1}) \log \|x''\| \frac{dH_{d-k}(x'')}{\sqrt{1 - \|x''\|^2}} \\ &= 2H_{k-1}(\mathbb{S}^{k-1}) \int_{\mathbb{B}_{d-k}} \left( \sqrt{1 - \|x''\|^2} \right)^{k-1} \log \|x''\| \frac{dH_{d-k}(x'')}{\sqrt{1 - \|x''\|^2}} \\ &= 2H_{k-1}(\mathbb{S}^{k-1}) \int_{\mathbb{B}_{d-k}} \left( \sqrt{1 - \|x''\|^2} \right)^{k-2} \log \|x''\| dH_{d-k}(x'') \\ &= 2H_{k-1}(\mathbb{S}^{k-1}) H_{d-k-1}(\mathbb{S}^{d-k-1}) \int_0^1 (1 - \rho^2)^{\frac{k-2}{2}} \rho^{d-k-1} \log \rho d\rho. \end{aligned} \quad (4.21)$$

Since  $\int_0^1 \rho^a \log \rho \, d\rho$  is finite when  $a > -1$ , it can be checked that the integral (4.21) is finite when  $1 \leq k < d$ . Thus, (4.18) holds when  $\mathbb{H}^d = \mathbb{R}^d$ .

The complex case  $\mathbb{H}^d = \mathbb{C}^d$  follows from the real case by identifying  $\mathbb{C}$  with  $\mathbb{R}^2$ . In particular, identify  $\mathbb{S}_{\mathbb{C}}^{d-1}$  with  $\mathbb{S}_{\mathbb{R}}^{2d-1}$ , identify  $G(k, \mathbb{C}^d)$  with  $G(2k, \mathbb{R}^{2d})$ , identify Borel sets in  $\mathbb{C}^d$  with Borel sets in  $\mathbb{R}^{2d}$ , and identify  $\sigma_{d-1}(E)$  with  $H_{2d-1}(E)/H_{2d-1}(\mathbb{S}^{2d-1})$ . With these identifications, in the complex case there holds

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \log(1 - \|P_W(x)\|^2) d\sigma_{d-1}(x) \\ &= \frac{2H_{2k-1}(\mathbb{S}^{2k-1})H_{2d-2k-1}(\mathbb{S}^{2d-2k-1})}{H_{2d-1}(\mathbb{S}^{2d-1})} \int_0^1 (1 - \rho^2)^{\frac{2k-2}{2}} \rho^{2d-2k-1} \log \rho \, d\rho, \end{aligned}$$

which is finite when  $1 \leq k < d$ .  $\square$

The next result is a corollary of Theorem 4.6 and Corollary 4.7 when  $X = \mathbb{S}^{d-1}$ ,  $Y = G(k, d)$ . Since a measure  $\mu \in \mathcal{M}(G(k, d))$  may be associated to a random subspace  $W$ , we shall say that  $\mu$  has Kaczmarz bounds  $\alpha_s$  and  $\alpha_{\log}$  if the associated random subspace  $W$  has these Kaczmarz bounds.

**Corollary 4.11.** *Let  $\mu \in \mathcal{M}(G(k, d))$ .*

- (1)  *$\mu$  is a minimizer of (4.1) if and only if the Kaczmarz bound  $\alpha_s$  is tight.*
- (2)  *$\mu$  is a minimizer of (4.2) if and only if the logarithmic Kaczmarz bound  $\alpha_{\log}$  is tight.*
- (3) *The invariant measure  $\Gamma = \Gamma_{k,d} \in G(k, d)$  minimizes both quantities (4.1) and (4.2).*

Corollary 4.11 shows that random subspaces chosen according to the invariant distribution on  $G(k, d)$  are minimizers for the problems (4.1) and (4.2). The next example shows that the minimizer for (4.1) is not necessarily unique; we shall further discuss uniqueness for (4.2) in Section 5.

**Example 4.12.** Let  $\{u_n\}_{n=1}^N \subset \mathbb{H}^d$  be a unit-norm tight frame for  $\mathbb{H}^d$  when  $d \geq 2$ . Namely, suppose that each  $u_n$  satisfies  $\|u_n\| = 1$  and that

$$\forall x \in \mathbb{R}^d, \quad \sum_{n=1}^N |\langle x, u_n \rangle|^2 = \frac{N}{d} \|x\|^2.$$

For examples of unit-norm tight frames see [9]. Let  $W \in G(1, d)$  be the random subspace defined by  $P[W = \text{span}(u_n)] = 1/N$  for each  $1 \leq n \leq N$ . Then,

$$\forall x \in \mathbb{S}^{d-1}, \quad \mathbb{E}(1 - \|P_W(x)\|^2) = 1 - \frac{1}{d} > 0.$$

In particular,  $W$  has a tight Kaczmarz bound of order  $s = 1$ , and by Corollary 4.11, the measure associated to  $W$  is a minimizer of (4.1) but is not the invariant measure on  $G(1, d)$ .

## 5. UNIQUENESS OF OPTIMAL DISTRIBUTIONS IN $G(1, d)$

This section addresses the uniqueness of the minimizers in (4.1) and (4.2) when the random subspaces  $W$  have dimension  $k = 1$ , i.e.,  $W \in G(1, d)$ . The main goal of this section is to show that the invariant distribution  $\Gamma_{1,d} \in \mathcal{M}(G(1, d))$  is the *unique* minimizer for the Kaczmarz problem (4.1) when  $0 < s < 1$ , and for the logarithmic Kaczmarz problem (4.2).

For perspective, Example 4.12 shows that minimizers for the Kaczmarz problem of order  $s = 1$  in (4.1) need not be unique.

The following theorem is the main result of this section.

**Theorem 5.1.** *Consider the problems of minimizing (4.1) and (4.2) for  $G(1, d)$ , i.e., when  $k = 1$ .*

- (1) *If  $k = 1$  and  $0 < s < 1$  then the invariant distribution  $\Gamma_{1,d} \in \mathcal{M}(G(1, d))$  is the unique minimizer of (4.1).*
- (2) *If  $k = 1$  then the invariant distribution  $\Gamma_{1,d} \in \mathcal{M}(G(1, d))$  is the unique minimizer of (4.2).*

The proof of Theorem 5.1 will proceed by identifying each subspace  $W \in G(1, d)$  with an element of  $\mathbb{H}^{d^2}$ , and then applying tools from potential theory. In particular, we shall make use of the following classical potential theoretic lemma. Note that this lemma involves *signed* measures. Recall that  $\int_E \int_E f(x, y) d\eta(x) d\eta(y)$  is absolutely integrable if

$$\int_E \int_E |f(x, y)| d|\eta|(x) d|\eta|(y) < \infty.$$

**Lemma 5.2.** *Suppose that  $E \subset \mathbb{H}^d$  is compact and that  $\eta$  is a signed Borel measure on  $E$  with zero total mass  $\int_E d\eta = 0$ .*

- (1) *If  $0 < s < 1$  and  $\int_E \int_E \|x - y\|^{2s} d\eta(x) d\eta(y) \geq 0$ , then the signed measure  $\eta$  is identically zero.*
- (2) *If  $\int_E \int_E \log \|x - y\| d\eta(x) d\eta(y) = 0$  and is absolutely integrable, then the signed measure  $\eta$  is identically zero.*

For the real case  $\mathbb{H}^d = \mathbb{R}^d$ , part (1) of Lemma 5.2 appears as Lemma 1 in [2], and part (2) appears as Lemma 3' in Section 33 of [14], cf. [16, 15]. The complex case  $\mathbb{H}^d = \mathbb{C}^d$  follows from the real case by identifying measures on  $\mathbb{C}^d$  with measures on  $\mathbb{R}^{2d}$ .

We are now ready to prove Theorem 5.1.

*Proof of Theorem 5.1. Step I.* We begin by identifying each subspace  $W \in G(1, d)$  with a unit-norm element of  $\mathbb{H}^{d^2}$ . Define the following set of  $d \times d$  matrices

$$\mathcal{E} = \{M \in \mathbb{H}^{d \times d} : M = xx^* \text{ for some } x \in \mathbb{S}^{d-1}\}.$$

With slight abuse of notation, we shall interchangeably think of  $\mathcal{E}$  as a subset  $\mathbb{H}^{d^2}$  and also as a set of  $d \times d$  matrices. Since the rank one projections  $xx^*$  have Frobenius norm one, the elements in  $\mathcal{E}$  are unit-norm when viewed as elements of  $\mathbb{H}^{d^2}$ , i.e.,  $\mathcal{E} \subset \mathbb{S}^{d^2-1}$ . It can also be verified that  $\mathcal{E}$  is a compact subset of  $\mathbb{H}^{d^2}$ .

Define the map  $\mathcal{L} : G(1, d) \rightarrow \mathcal{E}$  by  $\mathcal{L}(W) = xx^*$ , where  $x \in W \cap \mathbb{S}^{d-1}$  is any unit-norm element of  $W$ . This map does not depend on the choice of  $x$ , and so is well-defined. The map  $\mathcal{L}$  is an isometry when  $G(1, d)$  is endowed with the metric  $d_Y$  from the beginning of Section 4.2 and  $\mathcal{E}$  is endowed with norm metric on  $\mathbb{H}^{d^2}$ . The map  $\mathcal{L}$  is also bijective since  $xx^* = yy^*$  implies  $x = cy$  for some  $|c| = 1$ .



Given a Borel probability measure  $\mu \in \mathcal{M}(G(1, d))$ , the map  $\mathcal{L}$  induces a Borel probability measure  $\mathcal{L}_\mu \in \mathcal{M}(\mathcal{E})$  defined by

$$\forall E \in \mathcal{B}(\mathcal{E}), \quad \mathcal{L}_\mu(E) = \mu(\mathcal{L}^{-1}(E)),$$

where  $\mathcal{L}^{-1}(E) = \{W \in G(1, d) : \mathcal{L}(W) \in E\}$  is in  $\mathcal{B}(G(1, d))$  since  $\mathcal{L}$  is an isometry. Likewise, if  $S \in \mathcal{B}(G(1, d))$  then  $\mathcal{L}(S) = \{\mathcal{L}(s) : s \in S\} \in \mathcal{B}(\mathcal{E})$ . The bijectivity of  $\mathcal{L}$  guarantees a one-to-one correspondence between  $\mu$  and  $\mathcal{L}_\mu$  since

$$\forall S \in \mathcal{B}(G(1, d)), \quad \mu(S) = \mathcal{L}_\mu(\mathcal{L}(S)).$$

*Step II.* Next, define the map  $\Omega : \mathbb{S}^{d-1} \rightarrow G(1, d)$  by  $\Omega(u) = \text{span}(u)$ . The map  $\Omega$  is surjective, and  $\Omega(u) = \Omega(v)$  if and only if  $u = cv$  for some unimodular scalar  $c \in \mathbb{H}$  with  $|c| = 1$ . Let  $W \in G(1, d)$  and  $x \in \mathbb{S}^{d-1}$  be arbitrary, and pick any  $y \in \mathbb{S}^{d-1}$  such that  $W = \Omega(y)$ . Note that

$$\begin{aligned} \|\mathcal{L}(W) - \mathcal{L}(\Omega(x))\|^2 &= \|yy^* - xx^*\|_{\text{Frob}}^2 \\ &= \text{Tr}((yy^* - xx^*)^*(yy^* - xx^*)) \\ &= 2 - \text{Tr}(yy^*xx^*) - \text{Tr}(xx^*yy^*) \\ &= 2 - 2|\langle x, y \rangle|^2 \\ &= 2 - 2\|P_W(x)\|^2. \end{aligned} \tag{5.1}$$

*Step III.* To prove part (1) of Theorem 5.1, recall that the invariant measure  $\Gamma = \Gamma_{1,d} \in \mathcal{M}(G(1, d))$  is a minimizer of (4.1) by Corollary 4.7. Suppose that  $\nu \in \mathcal{M}(G(1, d))$  is also a minimizer of (4.1). By Theorem 4.6, there exists a constant  $C \in \mathbb{R}$  such that

$$\forall x \in \mathbb{S}^{d-1}, \quad \int_{G(1,d)} (1 - \|P_W(x)\|^2)^s d\nu(W) = \int_{G(1,d)} (1 - \|P_W(x)\|^2)^s d\Gamma(W) = C. \tag{5.2}$$

This will imply

$$\forall Z \in \mathcal{E}, \quad \int_{\mathcal{E}} \|Y - Z\|^{2s} d\mathcal{L}_\nu(Y) = \int_{\mathcal{E}} \|Y - Z\|^{2s} d\mathcal{L}_\Gamma(Y) = 2^s C. \tag{5.3}$$

Equation (5.3) is a result of (5.1), (5.2), and change of variables. Indeed, for any  $Z \in \mathcal{E}$ , there exists  $x \in \mathbb{S}^{d-1}$  such that  $\mathcal{L}(\Omega(x)) = Z$ , and

$$\begin{aligned} \int_{\mathcal{E}} \|Y - Z\|^{2s} d\mathcal{L}_\nu(Y) &= \int_{\mathcal{E}} \|Y - \mathcal{L}(\Omega(x))\|^{2s} d\mathcal{L}_\nu(Y) \\ &= \int_{\mathcal{L}^{-1}(\mathcal{E})} \|\mathcal{L}(W) - \mathcal{L}(\Omega(x))\|^{2s} d\nu(W) \\ &= 2^s \int_{G(1,d)} (1 - \|P_W(x)\|^2)^s d\nu(W) = 2^s C. \end{aligned} \tag{5.4}$$

An identical computation for  $\mathcal{L}_\Gamma$  yields (5.3).

Now define the signed measure  $\eta = \mathcal{L}_\nu - \mathcal{L}_\Gamma$  on the Borel subsets of  $\mathcal{E}$ . Since  $\mathcal{L}_\nu$  and  $\mathcal{L}_\Gamma$  are probability measures supported on  $\mathcal{E}$ ,  $\eta$  has zero total mass  $\int_{\mathcal{E}} d\eta = 0$ . Also, by (5.3),

$$\begin{aligned} \int_{\mathcal{E}} \int_{\mathcal{E}} \|Z - Y\|^{2s} d\eta(Y) d\eta(Z) &= \int_{\mathcal{E}} \left( \int_{\mathcal{E}} \|Z - Y\|^{2s} d\mathcal{L}_\nu(Y) - \int_{\mathcal{E}} \|Z - Y\|^{2s} d\mathcal{L}_\Gamma(Y) \right) d\eta(Z) \\ &= \int_{\mathcal{E}} (2^s C - 2^s C) d\eta(Z) = 0. \end{aligned} \quad (5.5)$$

By Lemma 5.2, equation (5.5) implies that the signed measure  $\eta$  is identically zero. Thus  $\mathcal{L}_\nu = \mathcal{L}_\Gamma$ . Since the correspondence between a measure  $\mu \in \mathcal{M}(G(1, d))$  and the measure  $\mathcal{L}_\mu \in \mathcal{M}(\mathcal{E})$  is bijective, it follows that  $\nu = \Gamma$ . This establishes part (1) of Theorem 5.1.

*Step IV.* The proof of part (2) of Theorem 5.1 proceeds similarly as Step III. Suppose that  $\nu \in \mathcal{M}(G(1, d))$  is also a minimizer of (4.2). By Theorem 4.6, there exists a constant  $C_1 \in \mathbb{R}$  such that

$$\forall x \in \mathbb{S}^{d-1}, \quad \int_{G(1, d)} \log(1 - \|P_W(x)\|^2) d\nu(W) = \int_{G(1, d)} \log(1 - \|P_W(x)\|^2) d\Gamma(W) = C_1. \quad (5.6)$$

This will imply

$$\forall Z \in \mathcal{E}, \quad \int_{\mathcal{E}} \log \|Y - Z\| d\mathcal{L}_\nu(Y) = \int_{\mathcal{E}} \log \|Y - Z\| d\mathcal{L}_\Gamma(Y) = 2^{-1}(C_1 + \log 2). \quad (5.7)$$

This is a similar argument as (5.4). For any  $Z \in \mathcal{E}$ , we find  $x$  such that  $\mathcal{L}(\Omega(x)) = Z$ , then (5.1) and (5.6) imply that

$$\begin{aligned} \int_{\mathcal{E}} \log \|Y - Z\| d\mathcal{L}_\nu(Y) &= \int_{\mathcal{E}} \log \|Y - \mathcal{L}(\Omega(x))\| d\mathcal{L}_\nu(Y) \\ &= \int_{\mathcal{L}^{-1}(\mathcal{E})} \log \|\mathcal{L}(W) - \mathcal{L}(\Omega(x))\| d\nu(W) \\ &= 2^{-1} \int_{G(1, d)} \log(2 - 2\|P_W(x)\|^2) d\nu(W) = 2^{-1}(\log 2 + C_1). \end{aligned} \quad (5.8)$$

An identical computation for  $\mathcal{L}_\Gamma$  yields (5.7).

Now define the signed measure  $\eta = \mathcal{L}_\nu - \mathcal{L}_\Gamma$  on the Borel subsets of  $\mathcal{E}$ . Since  $\mathcal{L}_\nu$  and  $\mathcal{L}_\Gamma$  are probability measures supported on  $\mathcal{E}$ ,  $\eta$  has zero total mass  $\int_{\mathcal{E}} d\eta = 0$ . Also, by (5.3),

$$\begin{aligned} \int_{\mathcal{E}} \int_{\mathcal{E}} \log \|Z - Y\| d\eta(Y) d\eta(Z) &= \int_{\mathcal{E}} \left( \int_{\mathcal{E}} \log \|Z - Y\| d\mathcal{L}_\nu(Y) - \int_{\mathcal{E}} \log \|Z - Y\| d\mathcal{L}_\Gamma(Y) \right) d\eta(Z) \\ &= \int_{\mathcal{E}} (2^{-1}(C_1 + \log 2) - 2^{-1}(C_1 + \log 2)) d\eta(Z) = 0. \end{aligned} \quad (5.9)$$

Note that  $\int_{\mathcal{E}} \int_{\mathcal{E}} \log \|Z - Y\| d\eta(Y) d\eta(Z)$  is absolutely integrable. To see this it suffices to show that  $\int_{\mathcal{E}} |\log \|Z - Y\|| d\mathcal{L}_\nu(Y) < \infty$  and  $\int_{\mathcal{E}} |\log \|Z - Y\|| d\mathcal{L}_\Gamma(Y) < \infty$ . A similar computation as in (5.8) shows that

$$\int_{\mathcal{E}} |\log \|Z - Y\|| d\mathcal{L}_\nu(Y) \leq 2^{-1} \left( \log 2 + \int_{G(1, d)} |\log(1 - \|P_W(x)\|^2)| d\nu(W) \right) = 2^{-1}(\log 2 - C_1).$$

An identical computation also shows that  $\int_{\mathcal{E}} |\log \|Z - Y\|| d\mathcal{L}_{\Gamma}(Y) < \infty$ .

By Lemma 5.2, equation (5.9) implies that the signed measure  $\eta$  is identically zero. Thus  $\mathcal{L}_{\nu} = \mathcal{L}_{\Gamma}$ . Since the correspondence between a measure  $\mu \in \mathcal{M}(G(1, d))$  and the measure  $\mathcal{L}_{\mu} \in \mathcal{M}(\mathcal{E})$  is bijective, it follows that  $\nu = \Gamma$ . This establishes part (2) of Theorem 5.1.  $\square$

We conclude this section with some questions. Theorem 5.1 shows that the invariant measure  $\Gamma_{k,d}$  is the unique minimizer of both (4.1) with  $0 < s < 1$  and (4.2) for subspaces of dimension  $k = 1$ , but it is not clear what happens for general values of  $k$ .

**Question 5.3.** *Is the invariant measure  $\Gamma_{k,d} \in \mathcal{M}(G(k, d))$  the unique minimizer of the problems (4.1) with  $0 < s < 1$  and (4.2) for each  $1 \leq k \leq d$ ?*

Example 4.12 shows that when  $s = 1$  the minimizers of (4.1) are not unique. It would be interesting to understand the issue of uniqueness in (4.1) when  $s > 1$ .

**Question 5.4.** *Is the invariant measure  $\Gamma_{k,d} \in \mathcal{M}(G(k, d))$  the unique minimizer of the problem (4.1) when  $s > 1$ ?*

## 6. NUMERICAL EXAMPLES

The numerical examples in this section plot error moments  $(\mathbb{E}\|x - x_n\|^{2s})^{1/s}$  of the subspace action algorithm (1.7) for different random subspace distributions and different values of  $s \geq 0$  (where  $\exp(\mathbb{E} \log \|x - x_n\|^2)$  corresponds to  $s = 0$ ). The expectations are approximated by averaging over 3000 trials in the first three examples, and by averaging over 9000 trials in the final example. All examples are done in real space  $\mathbb{R}^d$ , and the algorithm (1.7) is implemented with the initial estimate  $x_0 = 0$ .

**Example 6.1.** Figure 1 compares error moments for three different types of one-dimensional random subspaces in  $\mathbb{R}^2$ . Define the  $K$ th roots of unity frame  $\Phi_K = \{\varphi_k^K\}_{k=1}^K \subset \mathbb{R}^2$  by

$$\forall 1 \leq k \leq K, \quad \varphi_k^K = (\cos(2\pi k/K), \sin(2\pi k/K)). \quad (6.1)$$

It is well known that  $\Phi_K$  is a unit-norm tight frame for  $\mathbb{R}^2$  when  $K \geq 3$ , e.g., [1].

When  $K = 3$  and  $K = 5$  we consider the random subspace distributions that are defined as in Example 4.12 by randomly selecting at uniform the span of an element in  $\Phi_K$ . We also consider the invariant distribution  $\Gamma_{1,2}$ .

Note that the Kaczmarz bounds of  $\Gamma_{1,2}$  can be explicitly computed when  $s = 2, 1, 1/2, 0$ . When  $s = 0$ , Example 5.6 in [11] shows that  $\alpha_{\log} = 1/4$ . For  $s > 0$  and  $u \in \mathbb{S}^{d-1}$ ,

$$(\mathbb{E}(1 - \|P_W(u)\|^2)^s)^{1/s} = \left( \frac{1}{2\pi} \int_0^{2\pi} |\sin \theta|^{2s} d\theta \right)^{1/s},$$

so that  $\alpha_2 = \sqrt{3/8}$ ,  $\alpha_1 = 1/2$ , and  $\alpha_{1/2} = (2/\pi)^2$ .

The subplots in Figure 1 show error moments for the different values of  $s = 2, 1, 1/2, 0$  when  $x = (0.2296, 0.9361)$ . Each subplot plots error moments for each of the three random subspace distributions considered (randomized 3rd roots of unity, randomized 5th roots of unity, and the invariant distribution), and for comparison also plots the theoretical upper bounds  $\alpha_s^n$  from Theorem 3.2 and Theorem 3.3. Note that the uniform distribution has the smallest error among these three distribution and roughly coincides with  $\alpha_s^n$ . In the

case  $s = 1$ , all plots are very close to each other, which agrees with the fact that all three distributions have the same tight Kaczmarz bound  $\alpha_1 = 1/2$ .

**Example 6.2.** Figure 2 compares error moments for three different types of two-dimensional random subspaces in  $\mathbb{R}^5$ . Draw 5 two-dimensional subspaces  $W_n^5 \subset \mathbb{R}^5$ ,  $1 \leq n \leq 5$ , independently at random according to the uniform distribution on  $G(2, 5)$ . The resulting subspaces are now a deterministic collection. Let  $U$  be the random subspace defined by

$$\forall 1 \leq n \leq 5, \quad \Pr[U = W_n^5] = 1/5.$$

This will be referred to as the randomized 5 subspace distribution in this example. Similarly, draw 8 two-dimensional subspaces  $W_n^8 \subset \mathbb{R}^5$ ,  $1 \leq n \leq 8$ , independently at random according to the uniform distribution on  $G(2, 5)$ , and let  $V$  be the random subspace defined by

$$\forall 1 \leq n \leq 8, \quad \Pr[V = W_n^8] = 1/8.$$

This will be referred to as the randomized 8 subspace distribution in this example. See [3] for results concerning approximate tightness of randomly drawn fusion frames.

The subplots in Figure 2 show error moments for the different values of  $s = 2, 1, 1/2, 0$  for the randomized 5 subspace distribution, the randomized 8 subspace distribution, and the invariant distribution  $\Gamma_{2,5}$ . The signal  $x$  was taken to be  $x = (1, 1, 1, 1, 1)$ . In this experiment, the 8 subspace distribution outperforms the 5 subspace distribution, but the invariant distribution has the smallest error for each of the four values of  $s$ .

**Example 6.3.** Figure 3 illustrates Example 3.4 in  $\mathbb{R}^{100}$ , by comparing the invariant distribution  $\Gamma_{1,100}$  with the randomized orthonormal basis distribution (RONB) defined by (2.7). The figure compares error moments when  $s = 2, 1, 1/2$ . The signal  $x$  was taken to be  $x = (1, 1, 1, \dots, 1)$ . As expected from Example 3.4, the RONB distribution outperforms the invariant distribution when  $s = 1/2$  even though the invariant distribution has a smaller Kaczmarz bound of order  $1/2$ . When  $s = 1$  both distributions appear to give roughly the same error; this is consistent with the fact that both distributions have tight Kaczmarz bounds when  $s = 1$ , cf. Corollary 4.11 and Example 4.12. Finally, note that the invariant distribution outperforms the RONB distribution when  $s = 2$ .

**Example 6.4.** Figure 4 illustrates an example that is similar to Example 6.3, but with four-dimensional subspaces in  $\mathbb{R}^{100}$ . Let  $\{e_n\}_{n=1}^{100}$  be the canonical basis for  $\mathbb{R}^{100}$ , and for  $1 \leq i \leq 25$ , let  $W_i = \text{span}\{e_n\}_{n=4i-3}^{4i}$ . Note that  $\{W_i\}_{i=1}^{25}$  is a tight fusion frame for  $\mathbb{R}^{100}$  (with weights  $v_i = 1$ ). Let  $W$  be the random four-dimensional subspace defined by  $\Pr[W = W_i] = 1/25$ , and refer to this as the ONB distribution. The signal  $x$  was taken to be  $x = (1, 1, 1, \dots, 1)$ . Figure 4 compares the ONB distribution with the invariant distribution  $\Gamma_{4,100}$ . Similar to Example 6.3, the ONB distribution has smaller error moments than the invariant distribution when  $0 < s < 1$  (even though the invariant distribution uniquely achieves minimal Kaczmarz bounds when  $0 < s < 1$ ).

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## REFERENCES

- [1] J.J. Benedetto, A.M. Powell and Ö. Yılmaz. Sigma-Delta quantization and finite frames. *IEEE Transactions on Information Theory*, 52 (2006), no. 5, 1990–2005.
- [2] G. Björck. Distributions of positive mass, which maximize a certain generalized energy integral. *Arkiv för Matematik* 3.3 (1956): 255–269.
- [3] B.G. Bodmann. Random fusion frames are nearly equiangular and tight. *Linear Algebra and its Applications*, 439 (2013), no. 5, 1401–1414.
- [4] P. Casazza. The art of frame theory. *Taiwanese Journal of Mathematics*, 4 (2000), no. 2, 129–201.
- [5] J. Cahill, P. Casazza and S. Li. Non-orthogonal fusion frames and the sparsity of fusion frame operators. *Journal of Fourier Analysis and Applications*. 18 (2012), no. 2, 287–308.
- [6] P. Casazza and M. Fickus, Minimizing fusion frame potential. *Acta Applicandae Mathematicae*. 107 (2009), 7–24.
- [7] P. Casazza, M. Fickus, D. Mixon, Y. Wang, Z. Yang. Constructing tight fusion frames. *Applied and Computational Harmonic Analysis*, 30 (2011), no. 2, 175–187.
- [8] P. Casazza and G. Kutyniok. Frames of subspaces, pages 87–113 in *Wavelets, frames and operator theory* (editors C. Heil, P. Jorgensen and D. Larson), *Contemporary Mathematics*, 345, American Mathematical Society, Providence, RI, 2004.
- [9] P. Casazza and G. Kutyniok (editors). *Finite Frames, Theory and Applications*. Springer, New York, 2010.
- [10] P. Casazza, G. Kutyniok and S. Li. Fusion frames and distributed processing, *Applied and Computational Harmonic Analysis*, 25 (2008), no. 1, 114–132.
- [11] X. Chen and A.M. Powell. Almost sure convergence for the Kaczmarz algorithm with random measurements. *Journal of Fourier Analysis and Applications*, 18 (2012), no. 6, 1195–1214.
- [12] M. Ehler and K.A. Okoudjou. Minimization of the probabilistic  $p$ -frame potential. *Journal of Statistical Planning and Inference*, 142 (2012), no.3, 645–659.
- [13] T. Elfving. Block-iterative methods for consistent and inconsistent linear equations. *Numerische Mathematik*, 35 (1980), no. 1, 1–12.
- [14] O. Frostman. Potentiel d’équilibre et capacité des ensembles. *Comm. Sém. Math. Lund*, 3 (1935).
- [15] B. Fuglede. On the theory of potentials in locally compact spaces. *Acta Mathematica*, 103 (1960), no. 3, 139–215.
- [16] B. Fuglede. The logarithmic potential in higher dimensions. *Mat.-Fys. Medd. Danske Vid. Selsk.*, 33 (1960), no. 1, 14 pp.
- [17] L. Grafakos. *Classical and Modern Fourier Analysis*. Pearson Education, Inc., Upper Saddle River, New Jersey, 2004.
- [18] V.K. Goyal, J. Kovačević and J.A. Kelner. Quantized frame expansions with erasures. *Applied and Computational Harmonic Analysis*, 10 (2001), 203–233.
- [19] V.K. Goyal, J. Kovačević and M. Vetterli, Multiple description transform coding: Robustness to erasures using tight frame expansions, in *Proc. International Symposium on Information Theory (ISIT)*, 1998.
- [20] V.K. Goyal, J. Kovačević and M. Vetterli. Quantized frame expansions as source-channel codes for erasure channels, in *Proc. IEEE Data Compression Conference*, 1999, 325–355.
- [21] S. Kaczmarz. Angenäherte Auflösung von Systemen linearer Gleichungen. *Bulletin International de l’Académie Polonaise de Sciences A*, (1937), 355–357. English translation in: Approximate solution of systems of linear equations. *International Journal of Control* 57 (1993), 1269–1271.
- [22] A. Khosravi and M.S. Asgari. Frames of subspaces and approximation of inverse frame operator. *Houston Journal of Mathematics*, 33 (2007), 907–920.

- [23] P.G. Massey, M.A. Ruiz and D.Stojanoff. The structure of minimizers of the frame potential on fusion frames. *Journal of Fourier Analysis and Applications*, 16 (2010), no. 4, 514–543.
- [24] V.D. Milman and G. Schechtmann. *Asymptotic theory of finite-dimensional normed spaces*. Lecture Notes in Mathematics, Vol. 1200, Springer, New York, 1986.
- [25] D. Needell and J. Tropp. Paved with good intentions: analysis of a randomized block Kaczmarz method. *Linear Algebra and its Applications*, 441 (2014), 199–221.
- [26] D. Needell and R. Ward. Two-subspace projection method for coherent overdetermined systems. *Journal of Fourier Analysis and Applications*, 19 (2013), no. 2, 256–269.
- [27] P. Oswald. Stable space splittings and fusion frames, in *Proc. SPIE Vol. 7446, Wavelets XIII*, edited by V.K. Goyal, M. Papadakis, and D. van de Ville, 2009.
- [28] W. Rudin. *Real and Complex Analysis*, 3rd Edition. McGraw-Hill, New York, 1987.
- [29] A.N. Shiryaev. *Probability*, Second Edition. Springer, New York, 1989.
- [30] T. Strohmer and R. Vershynin. A randomized Kaczmarz algorithm with exponential convergence. *Journal of Fourier Analysis and Applications*, 15 (2009), no. 2, 262–278.

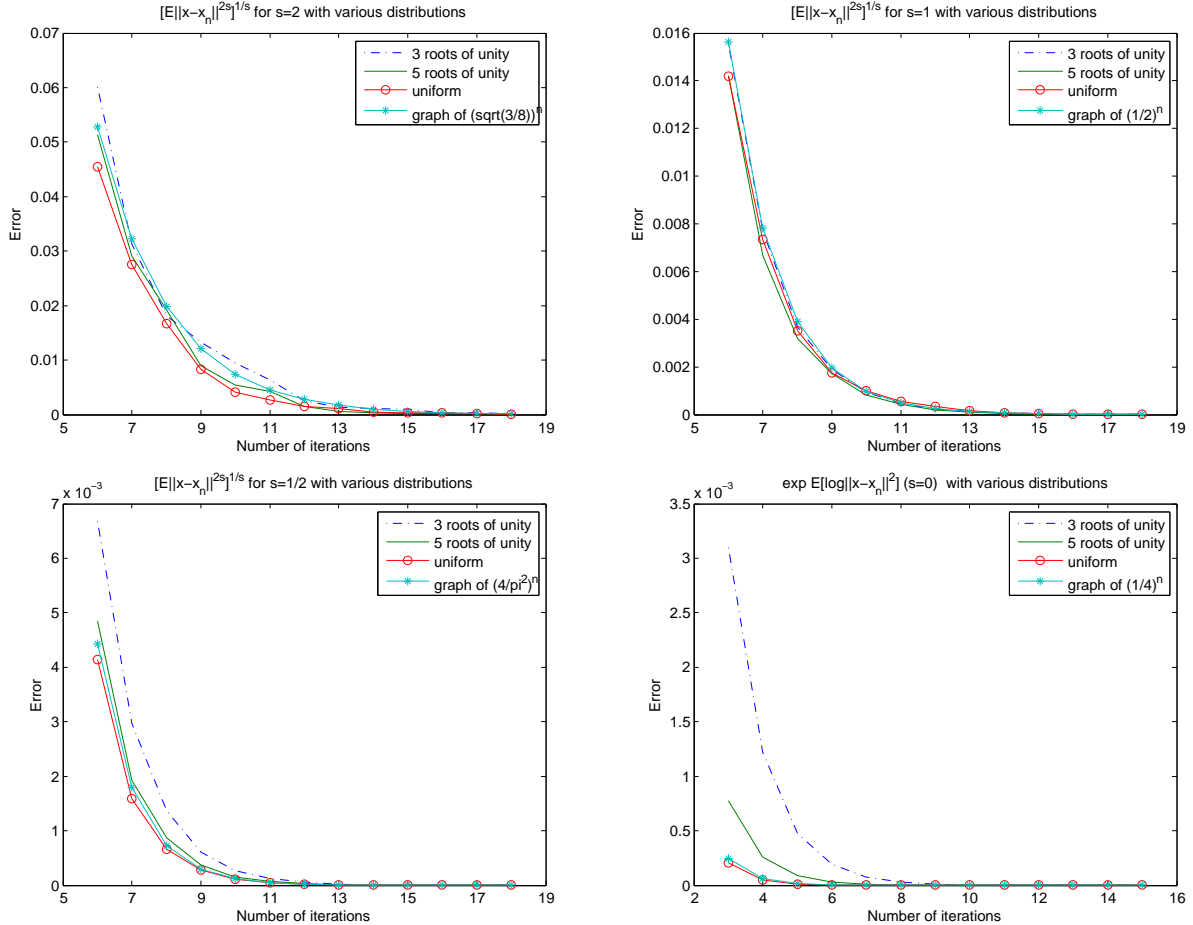


FIGURE 1. Error moments for the invariant distribution  $\Gamma_{1,2}$ , the randomized 3rd roots of unity distribution, and the randomized 5th roots of unity distribution, along with the Kaczmarz upper bounds for  $\Gamma_{1,2}$ , see Example 6.1.

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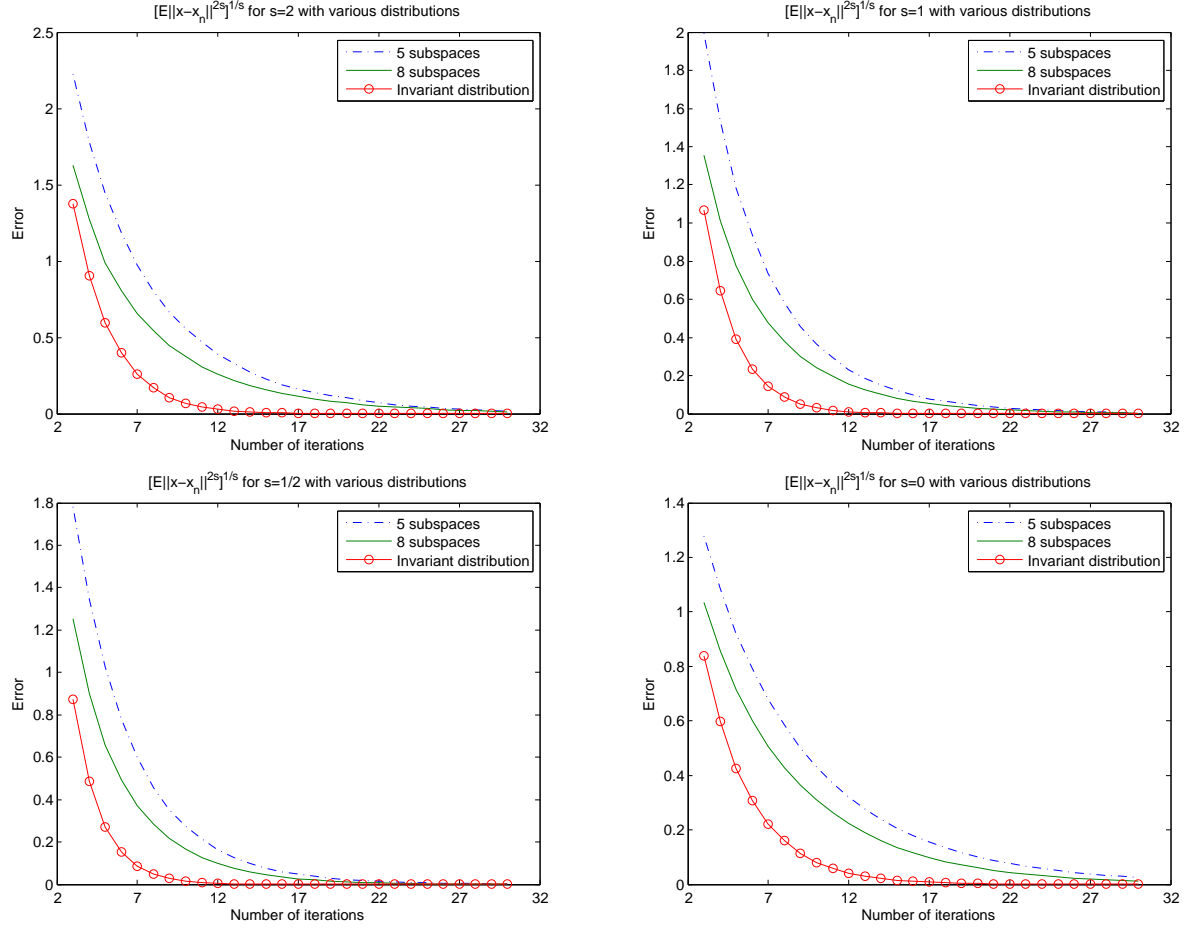


FIGURE 2. Error moments for the invariant distribution  $\Gamma_{2,5}$ , the randomized 5 subspace distribution, and the randomized 8 subspace distribution, see Example 6.2.

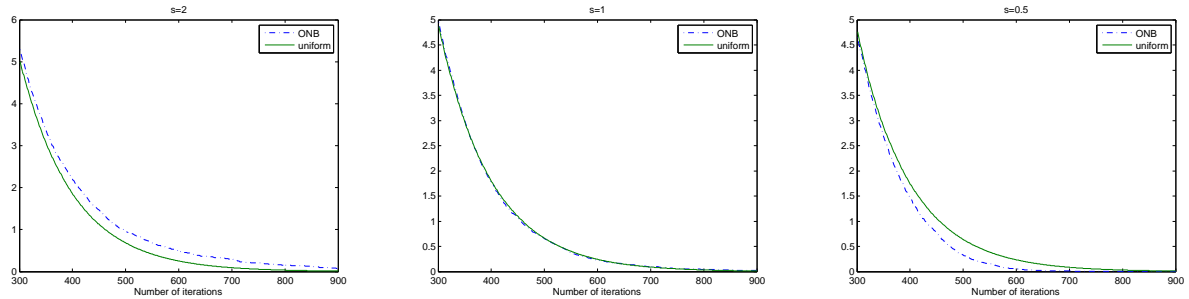


FIGURE 3. Error moments for the invariant distribution  $\Gamma_{1,100}$  and the RONS distribution, see Example 6.3.

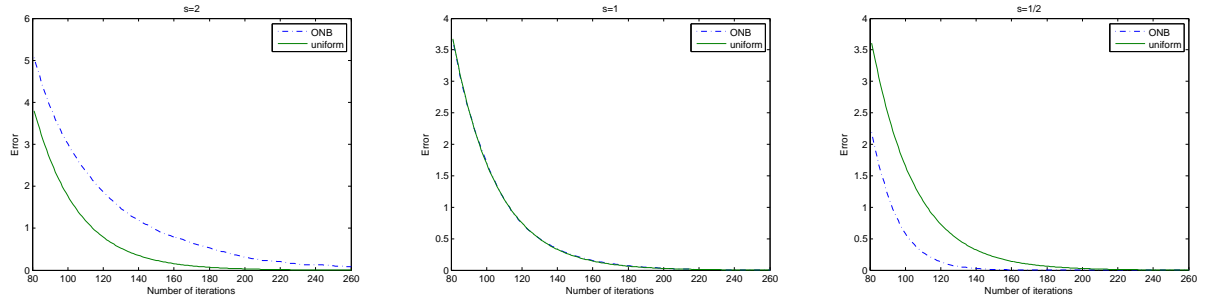


FIGURE 4. Error moments for the invariant distribution  $\Gamma_{4,100}$  and the fusion frame ONB distribution, see Example 6.4.